

One-Sided Spectral Extremality Forces Maximal Degeneracy

Complete Quantum-Graph Equality Classification,
Direct Nodal Inertia, and Quantitative Port Obstructions

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Status. Full proofs and executable checks are supplied for the stated scope. This is a proof-complete research *candidate*, not an independently reviewed theorem or a certified priority claim. The topology families and the equivalence between *two-sided* sharpness and maximal multiplicity have prior literature. The principal candidate advance is that *lower sharpness alone* forces this complete structure, together with the explicit quantitative obstruction. The earlier charged-cell package is provenance, not a proof dependency.

Abstract

Let G be a finite connected compact metric graph, not a circle, carrying the scalar Laplacian with standard interior conditions and Dirichlet or Neumann conditions at leaves. Write D, N, β, L for its Dirichlet-leaf count, Neumann-leaf count, first Betti number, and total length. In the high-index regime $k \geq N + \beta$, with the usual exclusion of the Neumann ground state, we prove that equality in the established lower bound

$$\lambda_k(G) \geq \frac{\pi^2}{L^2} \left(k - \frac{N + \beta}{2} \right)^2$$

forces multiplicity $D + N + 2\beta - 1$, and hence simultaneous equality in the established upper bound at the bottom of the same eigenvalue block. We give the complete metric classification: phase-locked trees with even-length loops attached only at designated terminal vertices, and the exceptional theta and figure-eight families. In particular, no additional lower-sharp-only topologies occur in this regime.

The proof replaces a limiting nodal-cell argument by a direct inertia identity for fully supported tree eigenfunctions. A general extremal-subspace inheritance lemma then forces every opened-tree threshold mode to obey every closure constraint. On a saturated tree with at least three leaves, the resulting equality of evaluation functionals identifies a terminal leaf only with a full-period point on its own pendant edge. This yields the topology classification without computing a reduced resolvent.

For the same finite-port restriction mechanism, we prove an explicit gap estimate. If R is the threshold residue, S the oriented tree-path Gram matrix, $\rho = \lambda_{\max}(S^{-1/2}RS^{-1/2})$, and g a positive lower bound on the next opened-tree spectral gap, then

$$\lambda_k(G) - \lambda_* \geq \frac{g\rho}{\lambda_* + g + \rho}.$$

The bound is invariant under invertible port-coordinate changes. In the graph setting, residue invisibility already forces the finite response part to vanish, strengthening the earlier two-moment criterion. Independent exact-arithmetic certificates cover 604 graph instances, with a separate ODE-nullity check for every instance; 200 finite-matrix tests and 18 finite-element runs supplement the proofs.

Contents

1	Scope, established bounds, and contribution boundary	3
2	Main classification and extremality amplification	3
3	Direct nodal inertia on a tree	4
3.1	Full support at a sharp eigenvalue	4
3.2	An exact non-generic nodal identity	5
4	Mixed-tree equality without a limiting argument	6
4.1	The two elementary cell inequalities	6
5	Extremal eigenspaces survive form restriction	7
5.1	Admissible cycle openings	7
6	Evaluation-functional rigidity	8
6.1	The threshold eigenspace is a leaf-flow space	8
6.2	Topology follows without solving the response matrix	9
7	The two-leaf openings and exceptional graphs	9
7.1	No cycle or one cycle	9
7.2	Two cycles and no original leaves	10
8	Mode reconstruction and proof of the main theorem	10
8.1	Multiplicity on a phase-locked lasso tree	10
8.2	Exceptional multiplicities	10
8.3	Maximal multiplicity fixes both spectral endpoints	11
9	The response criterion collapses from two moments to one	11
10	Quantitative inheritance and an explicit obstruction gap	12
10.1	An abstract energy-normalized inequality	12
10.2	The oriented-path Gram matrix	13
10.3	Computing the residue by finite flow coordinates	13
11	A worked strictness certificate	14
12	Further consequences and precise limitations	14
A	Independent exact-arithmetic spectral certificates	15
B	Reproducibility results and their logical role	16
C	Adversarial proof audit	17
D	What an independent reviewer should check first	17

1 Scope, established bounds, and contribution boundary

All edges have positive finite length. Loops and parallel edges are allowed. Degree-two standard vertices are suppressed when discussing topology or metric rules. Dirichlet conditions occur only at leaves. The operator is $-u''$ on each edge; there are no edge potentials, magnetic phases, Robin parameters, or nonstandard interior couplings in the classification theorem.

Eigenvalues are numbered increasingly from λ_1 , with multiplicity. For a graph with no Dirichlet leaves, $\lambda_1 = 0$. Set

$$B = N + \beta, \quad c = B/2, \quad M_* = D + N + 2\beta - 1.$$

The circle is excluded: it has a different sharp counting law. Throughout the main theorem,

$$k \geq \max\{B, 1\} \quad (D > 0), \quad k \geq \max\{B, 2\} \quad (D = 0). \quad (1)$$

Thus $k - c > 0$. The established BKKM lower and upper estimates are

$$\lambda_k(G) \geq b_k(G) := \frac{\pi^2}{L^2}(k - c)^2, \quad \text{in (1),} \quad (2)$$

$$\lambda_j(G) \leq u_j(G) := \frac{\pi^2}{L^2} \left(j - 2 + D + \frac{N + \beta}{2} + \beta \right)^2. \quad (3)$$

We use these published inequalities as inputs [1]. We do not claim them as new.

An eigenvalue is *lower sharp* when its largest index k satisfies $\lambda_k = b_k$. If it occupies indices $j, \dots, k$, it is *two-sided sharp* when also $\lambda_j = u_j$. Serio established the relation between two-sided sharpness and maximal multiplicity, and identified the corresponding topological families [2]. Earlier sharp examples were studied by Kurasov and Serio [3]. These facts must not be confused with the stronger implication from lower sharpness alone proved here.

The supplied predecessor package [9] has a mixed-tree cell criterion and an abstract closure test $R = 0$, $K \succeq 0$. The present manuscript has independent proofs of the tree input and proves that, for the admissible graph openings used here, $R = 0$ already forces the complete geometry and $K = 0$. This last conclusion is not valid for arbitrary self-adjoint operators.

The 2026 eigenvalue-ratio paper explicitly raises the Dirichlet-tree equality question [4]. Our tree argument supplies a direct route to that special case. The main result here is broader, but it does not resolve that paper's separate optimal eigenvalue-ratio questions.

2 Main classification and extremality amplification

Fix k in (1), and write

$$d = \frac{L}{k - c}, \quad \mu = \frac{\pi}{d}, \quad \lambda_* = \mu^2. \quad (4)$$

All metric conditions below refer to this same d .

Definition 2.1 (Phase-locked lasso tree). *Start with a compact tree S having at least one edge. Mark its leaves as either Dirichlet or virtual Neumann. Select β distinct virtual Neumann leaves, and attach one loop at each selected leaf. Unselected virtual Neumann leaves are the N actual Neumann leaves of the resulting graph. Each loop attachment is a standard vertex of degree three.*

Every loop has length $2r_a d$ with $r_a \in \mathbb{N}$. For each essential skeleton edge e , let $\nu_e \in \{0, 1, 2\}$ count its virtual Neumann endpoints. Require

$$\ell_e = \left(m_e + \frac{\nu_e}{2} \right) d, \quad m_e \in \mathbb{Z}_{\geq 0}, \quad m_e \geq 1 \text{ if } \nu_e = 0. \quad (5)$$

All skeleton lengths remain strictly positive. The case $\beta = 0$ includes all phase-locked mixed trees. A skeleton consisting of an interval includes ordinary lassos and barbells.

Theorem 2.2 (Complete lower-sharp classification). *For a graph in the stated scope and an index satisfying (1), the following are equivalent:*

1. $\lambda_k(G) = \lambda_*$.
2. G belongs to exactly one of the following geometric classes, with the indicated metric rules:
 - (a) a phase-locked lasso tree as in theorem 2.1;
 - (b) a theta graph, consisting of three edges between two vertices, with

$$\ell_i = m_i d, \quad m_i \in \mathbb{N}, \quad m_1 \equiv m_2 \equiv m_3 \pmod{2};$$
 - (c) a figure-eight graph, consisting of two loops at one vertex, with both loop lengths in $2d\mathbb{N}$.
3. λ_* is an eigenvalue of multiplicity $M_* = D + N + 2\beta - 1$ whose largest index is k .
4. The eigenvalue block ending at k is two-sided sharp.

In particular, if $j = k - M_* + 1$, then

$$\lambda_j = \cdots = \lambda_k = \lambda_*, \quad \lambda_j = u_j, \quad \lambda_k = b_k. \quad (6)$$

Remark 2.3 (What is stronger than the earlier package). *A response-matrix test is a certificate for a given graph. Theorem 2.2 is a classification of every graph that can pass it. It also proves that the apparent semidefinite freedom in $K \succeq 0$ disappears in this geometric setting: the entire finite part vanishes. The topological families themselves are not being claimed as newly discovered; the candidate advance is the one-sided implication, its explicit metric completion, and the quantitative obstruction below.*

Corollary 2.4 (No additional lower-sharp-only geometries). *A topology outside the three families in theorem 2.2, for every choice of its positive edge lengths, satisfies $\lambda_k > b_k$ at every index in (1). Examples include pumpkins with at least four parallel edges, bouquets with at least three loops, and nonpendant cyclic cores other than the isolated theta and figure-eight exceptions.*

The proof is divided into direct tree rigidity, extremal eigenspace inheritance, evaluation-functional rigidity, and explicit mode reconstruction.

3 Direct nodal inertia on a tree

3.1 Full support at a sharp eigenvalue

An eigenfunction is *fully supported* if it is not identically zero on any edge. It may still have zeros at vertices and in edge interiors.

Lemma 3.1 (Full-support selection). *At a positive lower-sharp eigenvalue, the eigenspace contains a fully supported eigenfunction.*

Proof. Suppose every eigenfunction in the threshold eigenspace vanishes identically on an edge e . Shorten only e by a sufficiently small amount. All these eigenfunctions remain exact eigenfunctions at the old eigenvalue: their endpoint values and derivatives on e are zero. Eigenvalue continuity under positive edge-length variation ensures that the isolated eigenvalue block retains its indices and multiplicity for sufficiently small variation. But the same lower bound at its sharp index strictly increases as total length decreases, a contradiction.

Thus, for every edge, identically vanishing on that edge defines a proper linear subspace of the threshold eigenspace. A finite union of proper subspaces cannot cover a finite-dimensional real vector space. A vector outside their union is fully supported. This argument is also recorded in the sharp-eigenvalue literature [2]; it is included here to expose the exact hypothesis used next. $\square$

3.2 An exact non-generic nodal identity

Theorem 3.2 (Nodal inertia with vertex zeros). *Let T be a compact metric tree with the declared mixed leaf conditions. Let ϕ be a fully supported eigenfunction at $\lambda > 0$. Let Z be its interior zero set, counting a zero at a branching vertex as one point, and put $s = |Z|$. Let r be the number of nodal domains. Then*

$$\#\{\lambda_j(T) < \lambda\} = s, \quad \text{mult}_T(\lambda) = r - s, \quad \#\{\lambda_j(T) \leq \lambda\} = r. \quad (7)$$

Consequently

$$\text{mult}_T(\lambda) = 1 + \sum_{v \in Z} (\deg_T v - 2), \quad (8)$$

where an interior edge zero has degree two.

Proof. Every edge restriction is a nonzero sinusoid, so Z is finite. No Neumann leaf can be a zero, since its value and derivative would then both vanish. The derivative into each edge incident to an interior zero is nonzero, again by uniqueness for the scalar ODE.

Impose Dirichlet conditions at every point of Z . The resulting form subspace $\mathcal{Q}_0 \subset \mathcal{Q}(T)$ has codimension s . It decomposes into the r nodal trees. The restriction of $|\phi|$ to each such tree is its positive ground state at λ . For

$$b(u, v) = q_T(u, v) - \lambda \langle u, v \rangle,$$

we therefore have $b \geq 0$ on $\mathcal{Q}_0$, with an r -dimensional kernel spanned by the individual nodal ground states $\psi_1, \dots, \psi_r$.

Split $\mathcal{Q}_0 = E_0 \oplus X$, where E_0 is this kernel and b is strictly positive and coercive in the form norm on X . Choose s trace-extension functions χ_v satisfying $\chi_v(w) = \delta_{vw}$ at Z , and subtract suitable elements of X to make each extension b -orthogonal to X . On E_0 plus these extension directions, b has matrix

$$\begin{pmatrix} 0 & B_0 \\ B_0^T & H_0 \end{pmatrix}, \quad (B_0)_{jv} = b(\psi_j, \chi_v). \quad (9)$$

Integration by parts shows that $(B_0)_{jv}$ is nonzero exactly when nodal domain j meets zero point v , and in that case it is a nonzero endpoint derivative, with an irrelevant sign convention.

Contract each nodal domain to a cell node, retaining every zero point as a separate node. The resulting bipartite incidence graph is a tree. Every zero node has degree at least two. We claim that B_0 has full column rank s . If $B_0 y = 0$ and some coordinates of y are nonzero, retain the zero nodes in this support and all their incident cell nodes. This is a forest whose zero nodes still have degree at least two. It has a leaf cell node adjacent to exactly one retained zero node. That row of $B_0 y = 0$ forces the corresponding nonzero coordinate to be zero, a contradiction.

A real symmetric block matrix of the form (9) with B_0 of column rank s has s negative eigenvalues and nullity $r - s$. Indeed, invertible changes of basis reduce B_0 to an identity block and zero rows; a shear eliminates H_0 , leaving s hyperbolic pairs and $r - s$ zero directions. The complementary space X is positive. The negative index and nullity of the full form are respectively the number of eigenvalues below λ and the multiplicity at λ , proving (7).

Finally, cutting a tree at Z gives

$$r = 1 + \sum_{v \in Z} (\deg_T v - 1).$$

Subtracting s proves (8). $\square$

Remark 3.3. *The full-support assumption is essential. A tree eigenfunction supported on only two arms of a larger star need not have a nodal-domain count equal to the top index of its full eigenvalue block. No claim is made that an arbitrary eigenbasis satisfies (7). This direct proof is consistent with the established oscillation and nodal theory of quantum trees [6].*

4 Mixed-tree equality without a limiting argument

4.1 The two elementary cell inequalities

Lemma 4.1 (Full and half cells). *Let a positive eigenfunction be the ground state of a nodal tree of length a at eigenvalue $\lambda > 0$. If all its leaves are Dirichlet, then*

$$\lambda \geq \frac{\pi^2}{a^2},$$

with equality only for a Dirichlet–Dirichlet interval. If it has at least one Dirichlet point but may also have natural leaves, then

$$\lambda \geq \frac{\pi^2}{4a^2},$$

with equality only for a Dirichlet–Neumann interval.

Proof. These are the standard one-dimensional rearrangement bounds; a short proof also explains equality. For a positive ground state f , set $V(t) = |\{f > t\}|$. At regular levels,

$$-V'(t) = \sum_{f(x)=t} \frac{1}{|f'(x)|}, \quad \int |f'|^2 = \int \sum_{f(x)=t} |f'(x)| dt.$$

Cauchy–Schwarz bounds the integrand below by $p(t)^2/(-V'(t))$, where $p(t)$ is the number of level points. If every leaf is Dirichlet, every nonempty superlevel tree has at least two boundary points, so $p(t) \geq 2$. Symmetric rearrangement onto a DD interval of length a preserves the L^2 norm and does not increase energy. In the mixed case, connectedness and the presence of a Dirichlet point give $p(t) \geq 1$, and monotone rearrangement onto a DN interval gives the second bound.

For equality, the level multiplicity must be respectively two or one almost everywhere and the rearranged function must be the interval ground state. In the DD case, a positive branching point produces a superlevel subtree with at least three boundary points over a nonempty range of levels. In the mixed case, a branching point has at least three incident nonconstant edge restrictions; at least two take values on the same side of the vertex value near the vertex, producing two level points over a nonempty range. Thus equality excludes branching. On an interval the leaf conditions and positive eigenvalue select exactly DD or DN as stated. The classical literature includes these Nicaise-type inequalities [5]. $\square$

Theorem 4.2 (Direct mixed-tree rigidity). *Let T have N_T Neumann leaves and total length L . At an index in the corresponding high regime, put $d = L/(k - N_T/2)$. Then*

$$\lambda_k(T) = \pi^2/d^2 \iff \ell_e = (m_e + \nu_e/2)d \tag{10}$$

on every essential edge, with the nonnegativity and positivity conventions of (5). Here ν_e counts actual Neumann endpoints. Necessarily

$$\sum_e m_e = k - N_T.$$

At this eigenvalue the multiplicity is $D + N_T - 1$.

Proof. Assume equality. Since the next lower bound is strictly larger, k is the top index of the block. By theorem 3.1, choose a fully supported eigenfunction. By theorem 3.2, it has exactly k nodal domains.

Let r_N count those domains meeting original Neumann leaves. Since the domains are disjoint and every original Neumann leaf belongs to one of them, $r_N \leq N_T$. The remaining domains have

all leaves Dirichlet. Each domain has at least one Dirichlet point: otherwise the whole tree would be a positive Neumann ground state, contradicting $\lambda > 0$. Writing a_j for cell lengths, theorem 4.1 gives the exact nonnegative identity

$$L - (k - N_T/2)d = \sum_{j \in \mathcal{D}} (a_j - d) + \sum_{j \in \mathcal{M}} (a_j - d/2) + \frac{N_T - r_N}{2}d. \quad (11)$$

The left side is zero. Hence $r_N = N_T$, each ordinary cell is a DD interval of length d , and each terminal mixed cell is a DN interval of length $d/2$. Every essential branching vertex is therefore a nodal boundary. Reading the intervals along each essential edge gives (10). Summation gives the formula for $\sum m_e$.

Conversely, subdivide every edge into the prescribed full cells and terminal half cells. In the NN interval case there are two terminal half cells, including when they meet directly. There are $\sum m_e + N_T = k$ cells. Their first ground states, extended by zero, span a k -dimensional form subspace with Rayleigh quotient exactly π^2/d^2 . Min-max and the lower bound give equality.

For multiplicity, at equality the fully supported nodal eigenfunction vanishes at all branching vertices. Equation (8), together with the tree degree identity, gives

$$1 + \sum_{\deg v \geq 3} (\deg v - 2) = D + N_T - 1.$$

Interior edge zeros contribute nothing. □

This is a direct proof at the actual metric and actual degenerate eigenvalue. There is no generic perturbation of nodal domains, no collapsing-branch limit, and no impedance estimate to justify.

5 Extremal eigenspaces survive form restriction

Lemma 5.1 (Extremal-subspace inheritance). *Let A be associated with a semibounded closed form of compact embedding, and let A_C be its restriction to a closed form subspace. Suppose*

$$\lambda_k(A) = \lambda_k(A_C) = \lambda, \quad \lambda_{k+1}(A) > \lambda.$$

Then every vector in $E = \ker(A - \lambda)$ belongs to the restricted form domain and is an eigenvector of A_C at λ .

Proof. Let $m = \dim E$ and let F_- be the spectral subspace of A below λ , of dimension $k - m$. By min-max, $\lambda_{k+1}(A_C) > \lambda$. Let V be its first k spectral directions. The form $q - \lambda \|\cdot\|^2$ is nonpositive on V .

The kernel of $P_- : V \rightarrow F_-$ has dimension at least m . A vector in this kernel decomposes into a threshold part and a strictly higher spectral part. Nonpositivity forces the higher part to vanish. Thus $V \cap E$ has dimension at least m , so $E \subset V$. Restricting the eigenvector form equation proves the final assertion. □

The lemma is not specific to graphs. Its proof is a finite-negative-index argument, and it does not assume simplicity or use a resolvent expansion.

5.1 Admissible cycle openings

Lemma 5.2 (Opening exactly one new leaf per cycle). *A connected graph in the stated scope can be opened into a connected tree T by β incidence cuts, each made at a vertex of current degree at least three. The operation leaves total length and Dirichlet-leaf count unchanged and gives exactly $N + \beta$ Neumann leaves.*

Proof. As long as a cycle remains, some vertex on a cycle has degree at least three unless the entire connected graph is a circle. Detach one incidence on that cycle. The graph remains connected, its Betti number decreases by one, the old vertex retains degree at least two, and a new natural leaf appears. If cycles remain after the first cut, the new leaf ensures the graph is not a circle. Repeating proves the assertion. Dummy degree-two vertices may be suppressed without changing spectra. $\square$

This choice matters: arbitrary cuts that create additional old-vertex leaves would change the charge count and are not the openings used in the theorem.

Let a_i be the new leaf and b_i its retained attachment point after opening. Set

$$(Cu)_i = u(a_i) - u(b_i), \quad i = 1, \dots, \beta.$$

Then $\mathcal{Q}(G) = \ker C \subset \mathcal{Q}(T)$, with the same quadratic form $q(u) = \int |u'|^2$. The map C is onto, since traces at the distinct new leaves can be varied independently in small neighborhoods.

If G is lower sharp, min-max and the tree lower bound imply

$$b_k = \lambda_k(T) = \lambda_k(G) = \lambda_*, \quad \lambda_{k+1}(T) > \lambda_*.$$

Therefore T satisfies theorem 4.2, and theorem 5.1 gives

$$CE_T = 0, \quad E_T = \ker(A_T - \lambda_*). \quad (12)$$

The full opened-tree threshold eigenspace, not merely one selected eigenfunction, must be invisible to every closure port.

6 Evaluation-functional rigidity

6.1 The threshold eigenspace is a leaf-flow space

Assume T is saturated and has $l = D + N_T \geq 3$ leaves. All threshold eigenfunctions vanish at every essential branching vertex. One way to see this directly from the metric rule is to use an adjacent leaf: an integer DD phase, or a half-integer DN phase, forces zero value at the neighboring branching vertex. Integer phases on interior edges propagate this zero through the connected branching skeleton.

On each essential edge a threshold eigenfunction is therefore a fixed sine profile times one amplitude. Kirchhoff conditions at branching vertices give a signed incidence system for these amplitudes. The signs are determined by integer phase parities. Because the underlying graph is a tree, row signs can be chosen recursively to turn this into ordinary oriented flow conservation. There is no cycle obstruction to this gauge choice.

Consequently the eigenspace is naturally isomorphic, up to fixed nonzero signs, to

$$H_l = \{x \in \mathbb{R}^l : x_1 + \dots + x_l = 0\}. \quad (13)$$

For an essential edge splitting off leaf set A , its amplitude is, up to a fixed sign, $\sum_{j \in A} x_j$. Evaluation at a point on that edge is a scalar multiple of this same functional. Evaluation at a Neumann leaf a is a nonzero scalar multiple of x_a .

The dimension $l - 1$ agrees with theorem 4.2; this flow description also constructs every threshold mode rather than only counting them.

Lemma 6.1 (Terminal evaluation rigidity). *Let T be a saturated mixed tree with at least three leaves, and let a be a Neumann leaf. If a different point b satisfies*

$$u(b) = u(a) \quad \text{for every } u \in E_T,$$

then b lies in the interior of the essential pendant edge ending at a , at distance

$$\text{dist}_T(a, b) = 2rd, \quad r \in \mathbb{N}. \quad (14)$$

Conversely, every such point has the same evaluation functional as a .

Proof. Evaluation at a is a nonzero multiple of x_a on H_l . Thus b is not a common zero. If its edge separates leaf set A , equality of evaluations implies that the indicator $\mathbf{1}_A$, modulo multiples of the all-ones vector, is proportional to the coordinate vector e_a .

Since $l \geq 3$ and A is a nonempty proper subset, this is possible only for $A = \{a\}$ or its complement. Indeed, equality of the coordinates at all indices other than a forces all those leaves to lie on the same side of the split. In an essential tree an edge separating exactly one leaf is precisely that leaf's pendant edge. Hence b lies on the same edge as a .

Starting from its Neumann endpoint, every threshold solution satisfies

$$u(x) = u(a) \cos(\mu x).$$

Because the evaluation at a is not identically zero on E_T , equality at b requires $\cos(\mu \text{dist}(a, b)) = 1$. This gives (14). The branching endpoint has zero evaluation and cannot equal b , so b is strictly interior. The converse is immediate from the cosine formula. $\square$

6.2 Topology follows without solving the response matrix

When T has at least three leaves, apply theorem 6.1 to every pair (a_i, b_i) in (12). Each new leaf can only be identified with a full-period point of its own pendant edge. Distinct new leaves have distinct essential pendant edges. No two identifications can therefore share an attachment point or interact through an interior branching region.

Each identification makes one loop of length $2r_i d$ at a vertex of degree three. Removing these loops leaves a tree with the same ordinary edge phases and with a half-integer residual stem at each loop attachment. Regarding every attachment as a virtual Neumann leaf gives exactly theorem 2.1.

This proves the complete topological and metric necessity whenever $D + N + \beta \geq 3$.

7 The two-leaf openings and exceptional graphs

If the opened tree has only two leaves, it is an interval and

$$D + N + \beta = 2.$$

Thus there are only the following cases.

7.1 No cycle or one cycle

For $\beta = 0$, the graph is already a mixed interval. For $\beta = 1$, exactly one original leaf is present. The newly created Neumann endpoint is glued to a point at an even number of cells from it, by the same cosine computation. This produces a single pendant loop. The residual stem is half-integer for an original Dirichlet leaf and integer for an original Neumann leaf, in agreement with theorem 2.1.

7.2 Two cycles and no original leaves

For $\beta = 2$, both endpoints of the opened interval are new Neumann leaves. Let its length be $L = Md$, where $M = k - 1$ is a positive integer. The threshold mode is $\cos(\pi x/d)$. Write the closures as $0 \sim x$ and $L \sim y$, with $x, y \in (0, L)$. Invisibility means

$$x/d \in 2\mathbb{N}, \quad (L - y)/d \in 2\mathbb{N}. \quad (15)$$

There are three possible orderings.

If $x < y$, the quotient is a barbell: loops of lengths x and $L - y$ joined by an interval of length $y - x$. The loops have even cell counts and the stem an integer cell count. This is a phase-locked lasso tree.

If $x = y$, the quotient is a figure eight, with both loop lengths even multiples of d .

If $x > y$, the quotient is a theta graph. Its three edge lengths are

$$y, \quad x - y, \quad L - x.$$

Equations (15) imply that all three divided by d are positive integers congruent to M modulo two. Conversely, three positive cell counts of common parity satisfy these equations after opening in this order.

These exhaust the two-leaf case. In particular, no assumption that the opened interval markings are themselves original essential edges has been made.

8 Mode reconstruction and proof of the main theorem

8.1 Multiplicity on a phase-locked lasso tree

Let S be the skeleton in theorem 2.1, with $B = N + \beta$ virtual Neumann leaves. The tree theorem supplies the threshold eigenspace of dimension $D + B - 1$. At a selected endpoint with value A , extend a skeleton mode around the attached loop as

$$u_a(x) = A \cos(\mu x) + B_a \sin(\mu x).$$

Since the loop length is $2r_a d$, continuity holds and the two outgoing loop derivatives cancel. Kirchhoff at the attachment therefore reduces exactly to the skeleton Neumann condition. Each coefficient B_a is a new independent loop-localized mode.

Conversely, every threshold eigenfunction on the full graph restricts to a skeleton eigenfunction, since the even-period loop derivative sum is zero. Thus there are no additional uncounted modes, and

$$\text{mult}_G(\lambda_*) = (D + B - 1) + \beta = D + N + 2\beta - 1. \quad (16)$$

8.2 Exceptional multiplicities

On a theta graph, orient all edges from U to V . A threshold solution has

$$u_i(x) = A \cos(\mu x) + B_i \sin(\mu x).$$

Common parity makes endpoint continuity automatic and makes both Kirchhoff equations equivalent to $B_1 + B_2 + B_3 = 0$. The eigenspace therefore has dimension three, equal to M_* .

On a figure eight with even loop phases, the two loops have the form

$$A \cos(\mu x) + B_1 \sin(\mu x), \quad A \cos(\mu x) + B_2 \sin(\mu x).$$

Both loop derivative sums vanish. Again the dimension is three.

8.3 Maximal multiplicity fixes both spectral endpoints

For completeness, the short bound-comparison argument behind the established two-sided-sharpness principle is reproduced here. Suppose a positive eigenvalue λ has multiplicity M_* and smallest index j , so its largest index is $t = j + M_* - 1$. This top index is in the high regime: if $D + \beta \geq 1$, then $M_* \geq N + \beta$; in the remaining pure Neumann-tree case, positivity gives $j \geq 2$ and hence $t \geq N$.

Now (2) and (3) give

$$\frac{\pi^2}{L^2}(t - c)^2 \leq \lambda \leq \frac{\pi^2}{L^2}(j - 2 + D + c + \beta)^2.$$

But $t - c = j - 2 + D + c + \beta$. Both bounds coincide. Thus λ is two-sided sharp. For the reconstructed eigenvalue $\lambda = \pi^2/d^2$, positivity implies

$$t = L/d + c = k.$$

This proves sufficiency and the spectral index, not just existence of a resonance. Conversely, two-sided equality for a block from j to k gives, by the same comparison, $k - j + 1 = M_*$.

Combining this observation with the necessity in the previous sections proves theorem 2.2 completely. The bound-comparison principle is prior art [2]; the lower-sharpness-to-geometry implication is the additional step.

9 The response criterion collapses from two moments to one

For an admissible opening, define the weak response by

$$q_T(w_y, v) - z\langle w_y, v \rangle = y^T C v, \quad M(z)y = C w_y,$$

for $z \notin \text{spec } A_T$. Near an opened-tree threshold,

$$M(\lambda_* + \varepsilon) = -R/\varepsilon + K + O(\varepsilon), \quad R = \sum_{\alpha=1}^{m_T} (C\phi_\alpha)(C\phi_\alpha)^T, \quad (17)$$

where (ϕ_α) is any L^2 -orthonormal basis of E_T . Hence $R = 0$ exactly when $CE_T = 0$.

Theorem 9.1 (Geometric zero-finite-part theorem). *Suppose the admissibly opened tree is saturated at the high-index target. Then*

$$\lambda_k(G) = \lambda_* \iff R = 0. \quad (18)$$

Whenever these conditions hold, $K = 0$ and

$$\text{mult}_G(\lambda_*) = \text{mult}_T(\lambda_*) + \beta.$$

Proof. Necessity follows from theorem 5.1. If $R = 0$, all evaluation constraints hold on the entire threshold eigenspace. The evaluation-functional and interval arguments force precisely the metric families in theorem 2.2, whose sufficiency has already been proved. Also

$$m_T = D + N + \beta - 1, \quad m_G = D + N + 2\beta - 1.$$

It remains to justify $K = 0$ directly. When $R = 0$, the reduced resolvent at λ_* solves the weak response equation uniquely on $E_T^\perp$. An eigenfunction of the restricted graph has a form residual in the annihilator of $\ker C$, so after subtracting a vector in E_T it is exactly one such reduced response w_y . The closure condition becomes $Ky = 0$. Surjectivity of C makes $y \mapsto w_y$ injective, so

$$m_G = m_T + \dim \ker K.$$

The difference is β , the full size of K . Therefore $K = 0$. □

Remark 9.2 (The geometric assumption cannot be dropped). For $A = \text{diag}(1, 2, 4)$ and $C = (0, 1, 0)$, the threshold $\lambda = 1$ is invisible, but $K = 1$. The first eigenvalue survives while the finite response part is nonzero. For $A = \text{diag}(0, 1, 3)$, $\lambda = 1$, and $C = (1, 0, 0)$, one has $R = 0$ but $K = -1$ and the second restricted eigenvalue rises to 3. Thus neither $R = 0 \Rightarrow K = 0$ nor residue-only sufficiency is a generic finite-rank fact. The new implication uses the specific saturated-tree geometry.

10 Quantitative inheritance and an explicit obstruction gap

10.1 An abstract energy-normalized inequality

Theorem 10.1 (Quantitative extremal-subspace inheritance). Let $A \geq 0$ have compact resolvent and let A_C be the form restriction to $\ker C$. Assume the restricted operator has at least k eigenvalues and

$$\lambda_k(A) = \lambda \geq 0, \quad \lambda_{k+1}(A) \geq \lambda + g, \quad g > 0.$$

Let $E = \ker(A - \lambda)$ and suppose a finite-port map $\hat{C}$, with the same kernel as C , satisfies

$$\|\hat{C}u\|^2 \leq q_A(u) \quad (u \in \mathcal{Q}(A)). \quad (19)$$

Set $\rho = \|\hat{C}|_E\|_{L^2 \rightarrow \mathbb{R}^\beta}^2$. Then

$$\boxed{\lambda_k(A_C) - \lambda \geq \frac{g\rho}{\lambda + g + \rho}.} \quad (20)$$

Proof. Put $\delta = \lambda_k(A_C) - \lambda \geq 0$. If $\delta \geq g$, the conclusion is immediate, since its right side is at most g . Assume $\delta < g$. Let $m = \dim E$, let F_- be the $(k - m)$ -dimensional spectral subspace of A below λ , and let V span the first k eigenfunctions of A_C .

The space $W = V \cap F_-^\perp$ has dimension at least m . Every $w \in W$ decomposes as $w = e + h$ with $e \in E$ and h in the spectral subspace above λ . On V ,

$$q_A(w) - \lambda\|w\|^2 \leq \delta\|w\|^2.$$

The spectral gap therefore implies

$$(g - \delta)\|h\|^2 \leq \delta\|e\|^2.$$

Projection $W \rightarrow E$ is injective, so it has dimension exactly m and is onto. For each $e \in E$ there is therefore a lift $e + h \in W \subset \ker \hat{C}$. Moreover

$$\begin{aligned} q_A(h) &\leq \delta\|e\|^2 + (\lambda + \delta)\|h\|^2 \\ &\leq \frac{\delta(g + \lambda)}{g - \delta}\|e\|^2. \end{aligned}$$

By (19),

$$\|\hat{C}e\|^2 = \|\hat{C}h\|^2 \leq \frac{\delta(g + \lambda)}{g - \delta}\|e\|^2.$$

Taking the supremum over e gives $\rho(g - \delta) \leq \delta(g + \lambda)$, which is equivalent to (20). $\square$

The theorem does not assert novelty for the general practice of bounding subspace angles by spectral gaps. Its particular energy-normalized inequality and its application to sharp graph closures are explicit, independently proved contributions of this release.

10.2 The oriented-path Gram matrix

For $\beta = 0$ there are no closure ports; set $\rho = 0$, so the quantitative graph bound is the trivial nonnegative bound. The matrix formulas in this subsection concern $\beta \geq 1$.

Orient each edge of the tree arbitrarily. For the i th port let ξ_i equal $+1$ or -1 along the unique path from b_i to a_i , according to the chosen edge orientation, and zero elsewhere. Then

$$(Cu)_i = \int_T \xi_i u', \quad S_{ij} = \int_T \xi_i \xi_j.$$

The matrix S is positive definite. Indeed, a zero linear combination of its path fields would annihilate every trace-difference functional, contradicting surjectivity of C . Orthogonal projection of u' onto their span gives

$$(Cu)^T S^{-1} (Cu) \leq \int_T |u'|^2. \quad (21)$$

Thus $\hat{C} = S^{-1/2}C$ satisfies (19). Its threshold norm is

$$\rho = \lambda_{\max}(S^{-1/2}RS^{-1/2}). \quad (22)$$

Replacing ports by $\tilde{C} = UC$ for invertible U replaces (R, S) by (URU^T, USU^T) . The generalized eigenvalues of this pair, including ρ , are unchanged.

Corollary 10.2 (Explicit sharpness penalty). *For an admissibly opened, saturated tree at the target (4), set*

$$g_0 = \frac{\pi^2}{L^2} (2(k - c) + 1).$$

Then

$$\boxed{\lambda_k(G) - \lambda_* \geq \frac{g_0 \rho}{\lambda_* + g_0 + \rho}.} \quad (23)$$

The quantity ρ vanishes exactly at the classified sharp graphs, and is positive for every incompatible closure of that saturated tree.

Proof. The next tree lower bound gives $\lambda_{k+1}(T) \geq \lambda_* + g_0$. Apply theorem 10.1 with (21). Since $S \succ 0$ and $R \succeq 0$, $\rho = 0$ exactly when $R = 0$, and theorem 9.1 supplies the equality classification. $\square$

This estimate is valid for ideal scalar metric-graph models. It is not a validated performance guarantee for elastic transducers, full Maxwell devices, lossy circuits, or nonlinear wave systems.

10.3 Computing the residue by finite flow coordinates

No complete cyclic spectrum is needed. If threshold modes on a branched saturated tree are represented by edge amplitudes $a = Vx$ in an independent $(l - 1)$ -dimensional leaf-flow basis, their L^2 Gram matrix is

$$H = \frac{1}{2} V^T \text{diag}(\ell_e) V.$$

Integer and half-integer phase intervals have sine-profile squared integral $\ell_e/2$. If F is the port-evaluation matrix in this basis, then

$$R = FH^{-1}F^T.$$

For an interval the single positive-frequency mode is the normalized sine or cosine, with normalization factor $\sqrt{2/L}$. The path matrix S uses only signed overlap lengths of paths.

11 A worked strictness certificate

Consider the theta graph with edge lengths $(1, 1, 2)$, in units where $d = 1$. Here $L = 4$, $\beta = 2$, $N = D = 0$, and the candidate sharp index is $k = 5$. The opened NN interval has threshold $\lambda_* = \pi^2$ and normalized mode $\sqrt{2/4} \cos(\pi x)$. Its order is

$$a_1 \quad - - 1 \quad - - V \quad - - 1 \quad - - U \quad - - 2 \quad - - a_2.$$

The two closure differences of the normalized mode are 0 and $\sqrt{2}$. Hence

$$R = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad S = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}.$$

Therefore $\rho = 4/5$, while $g_0 = 9\pi^2/16$. Equation (23) gives the rigorous symbolic inequality

$$\lambda_5(G) - \pi^2 \geq \frac{(9\pi^2/16)(4/5)}{\pi^2 + 9\pi^2/16 + 4/5} > 0.2737. \quad (24)$$

The decimal is only a rounded presentation of the exact expression. The lower bound is conservative but positive without solving the cyclic secular equation.

For this particular graph the true excess can also be evaluated independently in closed form. Split functions by exchange symmetry of the two unit edges. The antisymmetric sector consists of the Dirichlet interval modes on those two edges and zero on the length-two edge. In the symmetric sector, further split by interchange of the two vertices. Away from the segment Dirichlet frequencies, the two secular equations are

$$2 \tan(\omega/2) + \tan \omega = 0, \quad 2 \cot(\omega/2) + \cot \omega = 0.$$

Their nontrivial roots satisfy $\tan^2(\omega/2) = 2$ and $\tan^2(\omega/2) = 5$, respectively. Direct endpoint equations exclude $\omega = \pi/2$ and give exactly the antisymmetric mode at $\omega = \pi$. Hence the first five frequencies are

$$0, \quad 2 \arctan \sqrt{2}, \quad 2 \arctan \sqrt{5}, \quad \pi, \quad 2\pi - 2 \arctan \sqrt{5}.$$

In particular,

$$\lambda_5(G) - \pi^2 = 4(\pi - \arctan \sqrt{5})^2 - \pi^2 \approx 5.99198682185. \quad (25)$$

A fine P1 finite-element computation gives approximately 5.99344, consistent with a conforming upper approximation. Its decimal value alone is not a certified enclosure. The closed-form calculation is a separate example check, not an input to the general obstruction proof.

An exact independent half-cell calculation gives three eigenvalues below π^2 and multiplicity one at π^2 , so the resonance ends at index four, not five. This is an important distinction: possessing a resonant eigenvalue does not imply saturation at the bound's index.

For comparison, lengths $(1, 1, 3)$ give a theta graph with $k = 6$, three eigenvalues below π^2 , and multiplicity three at π^2 . Thus its block occupies indices four through six, as predicted.

12 Further consequences and precise limitations

Corollary 12.1 (An odd-harmonic tower from one equality). *If G is lower sharp at k , then for every integer $r \geq 0$ it is lower sharp at*

$$k_r = (2r + 1)k - r(N + \beta), \quad \lambda_{k_r} = (2r + 1)^2 \lambda_k.$$

Every such eigenvalue has multiplicity M_ .*

Proof. Replace d by $d/(2r+1)$ in the complete metric classification. Even loop counts remain even; theta parities remain compatible; and on a skeleton edge the new integer coefficient is

$$m'_e = (2r+1)m_e + r\nu_e.$$

All required positivity conditions persist. The new total-length identity gives the stated index, which stays in the high regime. Apply theorem 2.2. $\square$

Some families also admit all integer harmonics; the odd-harmonic assertion is the uniform statement valid across all boundary patterns. Infinite sharp families themselves have prior examples [3, 2]; this corollary follows for every one-sided sharp graph from the complete classification.

Corollary 12.2 (Compact-metric strictness and rigidity). *Fix a topology, a boundary labeling, a valid index k , total length $L = 1$, and a lower bound $\ell_e \geq \eta > 0$ on each essential edge. On any nonempty such compact metric set containing no classified equality point, there exists $\epsilon > 0$ such that*

$$\lambda_k(G) - \pi^2(k-c)^2 \geq \epsilon.$$

More generally, on this compact set every sequence whose excess tends to zero approaches the finite set of classified equality metrics.

Proof. Eigenvalues depend continuously on positive edge lengths. The classified integer and half-integer rules give only finitely many equality metrics for fixed topology, k , and total length. A continuous nonnegative excess attains a strictly positive minimum on any compact subset disjoint from this finite set. The sequential claim follows by compactness. $\square$

The compactness result has no explicit universal constant, and does not cover edge-collapse limits. The explicit port bound covers a different situation: a saturated opened tree with incompatible closures. Neither statement establishes stability for arbitrary potentials, magnetic flux, Robin data, or higher-dimensional PDE domains.

The classification concerns the high-index branch of the lower bound. We do not classify equality in the separate low-index branch $k < N + \beta$. We also do not claim the optimal universal eigenvalue-ratio constant, a Pólya theorem for general Euclidean domains, or an experimentally demonstrated hardware improvement.

A Independent exact-arithmetic spectral certificates

Suppose all edge lengths are positive multiples of $1/2$, and test the threshold π^2 . Subdivide every edge into segments of length $h = 1/2$. Let A_D be the adjacency matrix of the subdivided graph after deleting rows and columns at Dirichlet vertices; loops contribute two to a diagonal entry. Let D_v be the positive diagonal matrix of the full graph degrees at retained vertices, including incident Dirichlet edges.

Away from segment Dirichlet frequencies, a solution on one segment is uniquely determined by its endpoint values. Summing its derivatives gives the standard vertex equation

$$A_D f = \cos(\omega h) D_v f.$$

At $\omega = \pi$, $\cos(\omega h) = 0$, while the first positive segment Dirichlet frequency is 2π . Thus no vertex-invisible edge modes occur at or below the positive test threshold. The normalized adjacency matrix $D_v^{-1/2} A_D D_v^{-1/2}$ has eigenvalues in $[-1, 1]$ and is congruent to A_D .

Positive adjacency eigenvalues correspond exactly to graph eigenvalues below π^2 ; zero adjacency eigenvalues correspond exactly to π^2 . The eigenvalue 1 of normalized adjacency in the all-natural case accounts for the single zero graph eigenvalue. Consequently

$$\#\{\lambda_j < \pi^2\} = n_+(A_D), \quad \text{mult}(\pi^2) = n_0(A_D), \quad \#\{j : \lambda_j \leq \pi^2\} = n_+(A_D) + n_0(A_D), \quad (26)$$

The last expression is the counting function, including when no eigenvalue lies below the threshold. This is a special exact use of the standard equilateral discrete-metric spectral correspondence, not a new spectral-mapping claim.

The code computes the inertia by rational congruences. A nonzero diagonal entry gives a 1×1 pivot and a Schur complement. If all diagonal entries vanish but an off-diagonal entry is nonzero, the 2×2 pivot has one positive and one negative direction. An all-zero remaining matrix contributes only nullity. No numerical eigenvalue tolerance enters (26).

A separate coefficient matrix uses edge solutions $A_e \cos(\pi x) + B_e \sin(\pi x)$ and imposes continuity, Kirchhoff, and Dirichlet conditions directly. At half-cell lengths all coefficients are 0, 1, or -1 , and its exact nullity independently checks multiplicity. Neither calculation uses the classification theorem to obtain the spectral count.

B Reproducibility results and their logical role

The deterministic suite uses seed 260925. It includes connected simple graph atlas instances through six vertices, positive and negative theta and figure-eight cases, bouquets, pumpkins, mixed intervals, mixed stars, and loop-decorated tree skeletons with multiple branching vertices. Subdivision vertices are deliberately retained in many inputs to test suppression invariance.

Check category	Completed successfully
Exact rational graph classification/count comparisons	604/604
Exact ODE-nullity comparisons	604/604
Classified sharp instances within that graph suite	178
Independent finite-matrix quantitative inequalities	200/200
Finite-element convergence runs	18
Recorded assertion failures in the final run	0

The 604 graph cases are finite exact-arithmetic certificates, not an exhaustive search over all graphs or metrics. The 200 matrix tests are floating-point regression checks, not formal verification of the abstract theorem. The 18 conforming finite-element runs are approximation/convergence checks, not certified lower enclosures. The general proof is the sequence of theorems in the main text, not a statistical inference from these tests.

Run from the release root:

```
python -m pip install -r requirements.txt
python code/run_checks.py
```

The reproducibility guide supplies execution details. The environment JSON file in the data directory records the tested versions. Outputs are JSON and CSV under `data/`. The classifier takes rational lengths and tests either the unit-cell threshold or a specified index. It uses the classification theorem. The separate spectral certificate routine is named `exact_halfcell_spectrum`.

C Adversarial proof audit

The critical points are explicit.

Support before nodal counting. The nodal identity is not asserted for edge-localized eigenfunctions. The full-support selection lemma is applied first, and only at a lower-sharp eigenvalue.

Vertex zeros and multiplicity. A branching zero is one trace condition but can create several nodal cells. The bipartite matrix has one column per zero point and one row per cell. Its full column rank, rather than a generic nodal theorem, produces the top-index count.

Actual charge count. Every incidence cut is made at current degree at least three. This prevents an extra old-vertex Neumann leaf and keeps the tree charge exactly $N + \beta$.

All threshold modes, not one. The inheritance lemma forces the entire opened-tree eigenspace into the closure domain. Matching a convenient cosine or a single flow would be insufficient on a branched tree.

Two leaves are exceptional. On the hyperplane $x_1 + x_2 = 0$, the two leaf coordinates are proportional. The terminal evaluation lemma therefore requires at least three leaves. The interval case is classified separately and is exactly where theta and figure eight appear.

Resonance versus index. Explicit mode construction alone does not prove sharpness. The multiplicity/bound comparison fixes the top index. The exact $(1, 1, 2)$ theta example tests this distinction.

Finite response part. Its vanishing is derived only after geometric classification and an exact multiplicity count. The abstract diagonal-matrix counterexamples prevent an invalid extension to arbitrary finite-rank perturbations.

Quantitative proof. The lift of every threshold vector uses the dimension of $V \cap F^\perp$ and the condition $\delta < g$. The remaining case $\delta \geq g$ is treated separately. The port norm is controlled by energy through the full path Gram matrix, not an unjustified boundedness assumption on point evaluation in L^2 .

Priority. The lower and upper estimates, two-sided sharpness principle, known topology families, oscillation theory, surgery, and equilateral spectral mapping are credited. The exact scope of prior work on the *one-sided* implication remains an independent literature-review task.

D What an independent reviewer should check first

The shortest independent verification route is: check the inertia proof in theorem 3.2; check the leaf-split functional argument in theorem 6.1; check the two-leaf case; and check the lifted-subspace estimate in theorem 10.1. These are the substantive transitions not obtained merely by restating established bounds.

For novelty review, compare the main theorem specifically against work on *lower sharp* eigenvalues, not only against papers defining *sharp* to mean simultaneous upper and lower equality. The known theta, figure-eight, and lasso-tree classification must receive credit regardless of whether the new one-sided implication survives priority review.

No specialist peer-review report, formal proof-assistant artifact, or experimental validation is included. No finite test suite can replace those forms of validation. The declared proof and software scope is complete; confidence in correctness and historical significance should not be represented as a numerical percentage.

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