IUNO CALCULUS / APPLICATION NOTE

Counterexamples to Cartesian-product Negativity of Resistance Curvature

Exact memory, small certificates, and an infinite family

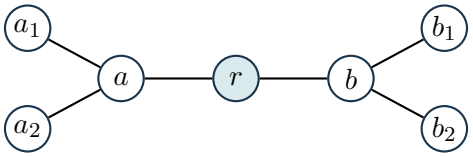
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Abstract

We give exact counterexamples to Conjecture 4 of the March 2024 version of *Node resistance curvature in Cartesian graph products* by Dawkins et al. A seven-vertex tree has a vertex of zero resistance curvature whose corresponding vertex in its Cartesian square has curvature $40/987 > 0$. A sixteen-vertex tree strengthens this: the factor curvature is $-1/2$, while the product curvature is $13/480 > 0$. Each calculation has a fifteen-entry rational potential certificate, verified against every vertex equation of the full product graph. We also prove an infinite-family theorem without relying on a numerical search: factor curvature can be arbitrarily negative while product curvature tends to $1/2$. The exact-memory and Kron-reduction branch of the supplied IUNO manuscript explains why a static reduction can conceal the mechanism.



The unweighted tree $T_{2,2}$: $p_T(r) = 0$, but $p_{T\square T}(r, r) = 40/987$.

Status	Exact counterexamples and complete proofs supplied; independent review pending.
Completeness	100% of the stated proof obligations are addressed in this note. This is a scope checklist, not a probability of correctness.
Verification	3/3 finite certificates pass exact checks; no floating-point assumption enters the proof.
Priority	No prior resolution located in the targeted search. Historical novelty is not certified.

1 The published target and the counterexample family

All graphs below are finite, connected, simple, and unweighted. Every edge has unit conductance. For vertices x, y, let $R_G(x, y)$ be effective resistance. The node resistance curvature is

$$p_G(x) = 1 - \frac{1}{2} \sum_{y \sim x} R_G(x, y). \tag{1}$$

The target is the pointwise implication proposed in Conjecture 4 of [1], in its arXiv v1 form:

$$p_G(x) \le 0, \quad p_H(y) \le 0 \quad \Longrightarrow \quad p_{G \square H}(x, y) < 0. \tag{2}$$

It is not a statement that all vertices of either factor must have nonpositive curvature. We address exactly the pointwise statement in (2).

Definition of the family. For integers $k \ge 2$ and $m \ge 1$, let $T_{k,m}$ have root r, hubs $h_1, \dots, h_k$, and leaves $\ell_{i,t}$ for $1 \le i \le k$, $1 \le t \le m$. Its edges are

$$r h_i \quad (1 \le i \le k), \qquad h_i \ell_{i,t} \quad (1 \le i \le k,\ 1 \le t \le m).$$

There are $n = 1 + k + km$ vertices. Every edge is a bridge, so its effective resistance is one. In particular,

$$p_{T_{k,m}}(r) = 1 - \frac{k}{2}. \tag{3}$$

Let $P_{k,m} = T_{k,m} \square T_{k,m}$ and $o = (r, r)$. Permuting branches in each coordinate and interchanging coordinates sends any edge incident to o to any other. Thus all $2k$ such edges have one common effective resistance $\rho_{k,m}$, and

$$p_{P_{k,m}}(o) = 1 - k\rho_{k,m}. \tag{4}$$

Theorem 1 (Two exact counterexamples)**.** *The following values hold:*

(k, m)	$\lvert V(T_{k,m})\rvert$	$p_T(r)$	$\rho_{k,m}$	$p_{T \square T}(o)$
$(2, 2)$	7	0	947/1974	40/987
$(3, 4)$	16	$-1/2$	467/1440	13/480

Consequently (2) *is false, including the variant with both hypotheses changed to strict negativity.*

The proof in Section 3 is an exact Kirchhoff calculation with small integer matrices. It does not depend on trusting a numerical eigensolver. Section 5 provides a separate analytic construction of counterexamples of unbounded size. No minimality claim is made for either finite example.

2 Electrical facts needed for verification

For a graph Laplacian L, the potential v for one unit of current entering at a and leaving at b satisfies $Lv = e_a - e_b$. Once $v_b = 0$ is imposed, the solution is unique and $R_G(a, b) = v_a$. Indeed, the kernel of L consists exactly of constant vectors, since

$$v^T L v = \sum_{\{x,y\} \in E} (v_x - v_y)^2.$$

The energy of this unit-current solution is also its voltage difference. Equivalently, effective resistance is the minimum of $\sum_e f_e^2$ over unit flows from a to b. The latter statement follows by decomposing any such flow into the gradient solution and a divergence-free flow: their energy cross term vanishes by summation by parts. These facts will justify both the certificates and the comparison argument.

3 A fifteen-entry proof certificate

Inject one unit of current at (h_1, r) and remove it at $o = (r, r)$. The symmetries fixing both terminals give five first-coordinate classes and three second-coordinate classes:

$$\mathcal{A} = (\{r\}, \{h_1\}, \{\ell_{1,t}\}_t, \{h_i : i > 1\}, \{\ell_{i,t} : i > 1\}),$$
$$\mathcal{B} = (\{r\}, \{h_i\}_i, \{\ell_{i,t}\}_{i,t}).$$

The potential is constant on each of the fifteen product classes, by uniqueness. Let V be the corresponding 5×3 matrix. The per-vertex Laplacian matrices on the two class systems are

$$Q_{k,m} = \begin{pmatrix} k & -1 & 0 & -(k-1) & 0 \\ -1 & m+1 & -m & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & m+1 & -m \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \qquad H_{k,m} = \begin{pmatrix} k & -k & 0 \\ -1 & m+1 & -m \\ 0 & -1 & 1 \end{pmatrix}. \tag{5}$$

They need not be symmetric: entries count neighbours of an individual vertex in each class. We use them as exact equations on invariant functions, not as unweighted graph Laplacians.

Write J for the 5×3 matrix with $J_{2,1} = 1$, $J_{1,1} = -1$, and every other entry zero (indices here start at one). Kirchhoff's equations become

$$Q_{k,m}V + VH_{k,m}^T = J, \qquad V_{1,1} = 0. \tag{6}$$

Both nonzero current classes are singletons. Thus no orbit-size factor is missing from the right-hand side.

Seven-vertex certificate. For $(k, m) = (2, 2)$, take $V = Z/7896$, where

$$Z = \begin{pmatrix} 0 & 1504 & 1786 \\ 3788 & 2520 & 2305 \\ 3002 & 2609 & 2457 \\ 1100 & 1428 & 1549 \\ 1322 & 1433 & 1491 \end{pmatrix}.$$

Direct integer multiplication gives $Q_{2,2}Z + ZH_{2,2}^T = 7896J$. For example, the root equation is $-3788 - 1100 - 2(1504) = -7896$; the source equation is $5(3788) - 2(3002) - 2(2520) = 7896$. All other entries are zero. Consequently

$$\rho_{2,2} = V_{2,1} = \frac{3788}{7896} = \frac{947}{1974}, \qquad p_{P_{2,2}}(o) = 1 - 2\frac{947}{1974} = \frac{40}{987} > 0.$$

Strictly negative factors. For $(k, m) = (3, 4)$, take $V = Z/4320$, where

$$Z = \begin{pmatrix} 0 & 615 & 690 \\ 1401 & 912 & 859 \\ 1038 & 917 & 888 \\ 537 & 624 & 643 \\ 606 & 629 & 636 \end{pmatrix}.$$

Here $Q_{3,4}Z + ZH_{3,4}^T = 4320J$. Thus

$$\rho_{3,4} = \frac{1401}{4320} = \frac{467}{1440}, \qquad p_{P_{3,4}}(o) = 1 - 3\frac{467}{1440} = \frac{13}{480} > 0.$$

Equation (3) gives $p_{T_{3,4}}(r) = -1/2$. This proves Theorem 1. The accompanying verifier also lifts these potentials and checks Kirchhoff's equation at every original product vertex.

4 The IUNO connection: the information that static reduction loses

The supplied IUNO manuscript's Sections 8–9 retain eliminated graph nodes as an exact memory kernel and identify Kron reduction with its integrated mass [2]. This is directly applicable to $T_{k,m}$.

Retain the root and hubs, $B = \{r, h_1, \dots, h_k\}$, and eliminate all leaves, $I = \{\ell_{i,t}\}$. Let $L_\star$ be the unweighted Laplacian of $K_{1,k}$ on B, and let $P_H = \mathrm{diag}(0, 1, \dots, 1)$ project onto the hubs. Then

$$L_{II} = I_{km}, \qquad L_{BB} = L_\star + mP_H, \qquad L_{BI}L_{IB} = mP_H.$$

IUNO's exact heat-memory kernel becomes

$$K_m(t) = L_{BI}e^{-tL_{II}}L_{IB} = me^{-t}P_H. \tag{7}$$

It has rank k for $m > 0$: the km leaf coordinates have only k dynamically observable aggregate channels at this boundary. The remaining leaf-difference modes are invisible to it. The zero-frequency reduction is

$$L_K = L_{BB} - \int_0^\infty K_m(t)\,dt = L_\star. \tag{8}$$

Thus every m has exactly the same static boundary network. But the frequency-dependent boundary operator is

$$\begin{aligned} S_m(s) &= sI + L_{BB} - L_{BI}(sI + L_{II})^{-1}L_{IB} \\ &= sI + L_\star + \frac{ms}{s+1}P_H, \qquad s > 0. \end{aligned} \tag{9}$$

The parameter m disappears at $s = 0$ and remains present at every positive frequency. In a Cartesian product the Laplacian is a Kronecker sum. Diagonalizing one factor evaluates shifted resolvents of the other factor at its positive eigenvalues. Hence retaining only (8) before forming a product loses precisely the frequency dependence displayed in (9).

Proposition 2 (A sign cannot be inferred from the static reduction)**.** *At $k = 2$, the trees $T_{2,1}$ and $T_{2,2}$ both reduce to $K_{1,2}$ by eliminating leaves, and both roots have curvature zero. Nevertheless,*

$$p_{T_{2,1}\square T_{2,1}}(o) = -\frac{27}{550} < 0, \qquad p_{T_{2,2}\square T_{2,2}}(o) = \frac{40}{987} > 0.$$

Proof. The reductions and factor curvatures follow from (3) and (8). For the first product, (6) is satisfied by

$$V = \frac{1}{2200}\begin{pmatrix} 0 & 396 & 484 \\ 1154 & 750 & 653 \\ 916 & 797 & 725 \\ 254 & 350 & 403 \\ 316 & 347 & 375 \end{pmatrix}.$$

Multiplication gives $Q_{2,1}Z + ZH_{2,1}^T = 2200J$. The edge resistance is $1154/2200 = 577/1100$, giving the asserted curvature; the second value was proved above. □

Attribution boundary. Schur complementation, electrical comparison, symmetry quotients, and exact linear memory are classical. The use of IUNO here is its organized instruction to retain the eliminated influence and inspect its frequency dependence. The proposed new results are the counterexamples and family theorems, not a claim to have invented these classical tools.

5 An analytic infinite-family theorem

Theorem 3 (Arbitrarily negative factors, positive product)**.** *Fix $k \geq 2$. Then*

$$\lim_{m\to\infty} p_{T_{k,m}\square T_{k,m}}(o) = \frac{1}{2}, \qquad p_{T_{k,m}}(r) = 1 - \frac{k}{2}.$$

More quantitatively, if $q = \lfloor m/k \rfloor \geq 1$, then

$$\frac{1}{2} - \frac{k-1}{1+kq/6} - \frac{1}{2(1+kq/3)} \;\leq\; p_{T_{k,m}\square T_{k,m}}(o) \;\leq\; \frac{1}{2}. \tag{10}$$

In particular, for $m \geq 12k$,

$$p_{T_{k,m}\square T_{k,m}}(o) \geq \frac{5k+1}{(2k+1)(4k+1)} > 0. \tag{11}$$

Proof. We construct a subnetwork of the Cartesian square. Write

$$A_i = (h_i, r), \qquad B_j = (r, h_j), \qquad C_{ij} = (h_i, h_j).$$

Keep all $2k$ unit edges from o to the A_i and B_j. At every hub h_i, allocate q different leaves to each of the k possible indices j; this is possible since $kq \leq m$. For each (i, j), retain q paths

$$A_i \;--\; (\ell_{i,t}, r) \;--\; (\ell_{i,t}, h_j) \;--\; C_{ij},$$

using the leaves allocated to j. Likewise retain q paths

$$B_j \;--\; (r, \ell_{j,u}) \;--\; (h_i, \ell_{j,u}) \;--\; C_{ij},$$

using leaves allocated to i at hub h_j. Distinct allocations ensure that the interiors of these paths are disjoint. Interiors of the first kind have a leaf in the first coordinate; those of the second kind have a leaf in the second coordinate, so the two kinds do not collide. Delete every other edge and every unused vertex.

Each path has resistance three. Parallel and series reduction therefore replaces the paths through C_{ij} by an edge A_iB_j of resistance $6/q$. The resulting network has the root joined by unit edges to every vertex of a weighted $K_{k,k}$, whose edges have conductance $w = q/6$.

Ground o in this comparison network. Its remaining Laplacian is $M = I_{2k} + wL(K_{k,k})$. On the constant subspace, the part-contrast subspace, and the subspace of vectors summing to zero separately in each part, the eigenvalues of $L(K_{k,k})$ are respectively 0, $2k$, and k. The corresponding squared projection lengths of a coordinate vector are $1/(2k)$, $1/(2k)$, and $(k-1)/k$. Its effective resistance to the root is thus

$$\widehat{\rho} = \frac{1}{2k} + \frac{k-1}{k(1+kq/6)} + \frac{1}{2k(1+kq/3)}. \tag{12}$$

Adding back deleted edges and vertices cannot increase effective resistance: a unit flow in the subnetwork is also a unit flow in the original network, with zero current on added edges. The energy minimum can only decrease. Therefore $\rho_{k,m} \leq \widehat{\rho}$, and substitution into (4) gives the lower bound in (10).

For the upper bound, every unit flow from o has total signed current one across its $2k$ incident edges. Cauchy–Schwarz gives energy at least $1/(2k)$ on those edges alone. Thus $\rho_{k,m} \geq 1/(2k)$ and $p_{P_{k,m}}(o) \leq 1/2$. As $m \to \infty$, $q \to \infty$ and the lower bound tends to $1/2$, proving the limit. If $q \geq 12$, the lower bound is at least

$$\frac{1}{2} - \frac{k-1}{2k+1} - \frac{1}{2(4k+1)} = \frac{5k+1}{(2k+1)(4k+1)} > 0.$$

Finally, (3) supplies the factor curvature. □

6 Consequences, scope, and reproducibility

Proposition 4 (Small-parameter infinite families)**.** *For every $m \geq 2$, $T_{2,m}$ has root curvature zero and its Cartesian square has root curvature at least* $40/987$*. For every $m \geq 4$, $T_{3,m}$ has root curvature* $-1/2$ *and its Cartesian square has root curvature at least* $13/480$*.*

Proof. Increasing m embeds the smaller tree and its Cartesian square as subgraphs. The product root keeps exactly the same incident edges. Each resistance across those edges is nonincreasing by the unit-flow comparison argument, so the root curvature is nondecreasing. Apply Theorem 1. □

What has been resolved. The explicit pointwise implication (2), as written in [1], is disproved. Both equality in the hypotheses and strict negativity are covered. The examples are finite, simple, connected, unweighted trees; no change of graph product or curvature convention is used. The sign reversal persists in infinite families. Theorem 3 further shows that there is no rescue obtained merely by requiring arbitrarily large negative factor curvature.

What is not claimed. The examples are not asserted to have minimum order. Classification under changes of edge weights and the source's rectangular grid theorem are outside this work. Historical priority and independent acceptance remain separate questions.

Exact verification. Run `python verify.py` using Python 3.9 or newer; no third-party package is needed. It checks (i) the integer identities (6), (ii) the lifted Kirchhoff equation at every vertex of the 25-, 49-, and 256-vertex product graphs, and (iii) all stated finite resistance and curvature fractions. It also performs an independent rational Gaussian solve for the seven-vertex example's 49-vertex product. Integer residual verification suffices for the larger example because the grounded solution is unique. The JSON certificate file makes every input explicit. Passing these checks is exact arithmetic verification, not formalization in a proof assistant and not independent peer review.

Discovery versus proof. Numerical spectral exploration located a larger candidate; reduction and structured tree exploration produced these examples. The proof relies on the displayed identities, uniqueness of potentials, and analytic comparison, not on the search or its numerical tolerances.

Status and completeness. All finite proof obligations are complete (3/3 cases, or 100% of the specified certificate checklist). The family proofs are supplied in full. Independent expert reviews completed: 0. These counts are not correctness probabilities.

Literature check, 27 September 2026. The inspected arXiv version states the target as Conjecture 4. Searches by title, conjecture wording, counterexample terminology, and the resulting fractions did not locate a prior resolution. The search is nonexhaustive. Attribution is to the exact arXiv statement; the journal full text was not accessible.

References

[1] A. Dawkins, V. Gupta, M. Kempton, W. Linz, J. Quail, H. Richman, and Z. Stier. *Node resistance curvature in Cartesian graph products.* arXiv:2403.01037v1, 2 March 2024, Conjecture 4. https://arxiv.org/abs/2403.01037.

[2] *IUNO Calculus: Renewal Network Geometry*, version 2.0.0, 27 September 2026. User-supplied manuscript, Sections 8–9, especially Theorems 8.1 and 9.1. This note restates every part required for its arguments.